Nonregular Languages

Theorem: The following are all equivalent:

- L is a regular language.
- There is a **DFA** D such that $\mathscr{L}(D) = L$.
- There is an **NFA** N such that $\mathscr{L}(N) = L$.
- There is a **regular expression** R such that $\mathscr{L}(R) = L$.

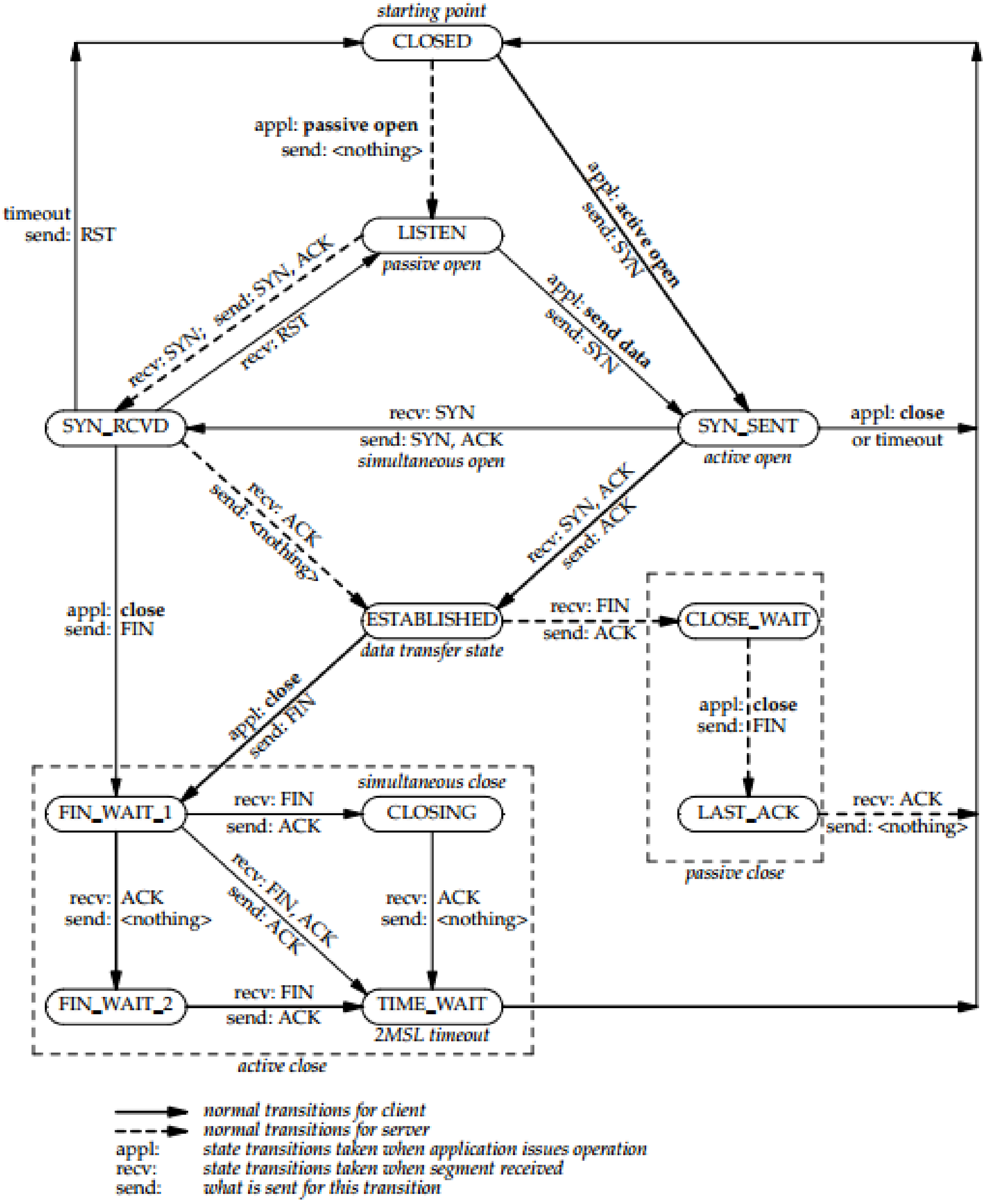
New Stuff!

Why does this matter?

Buttons as Finite-State Machines:

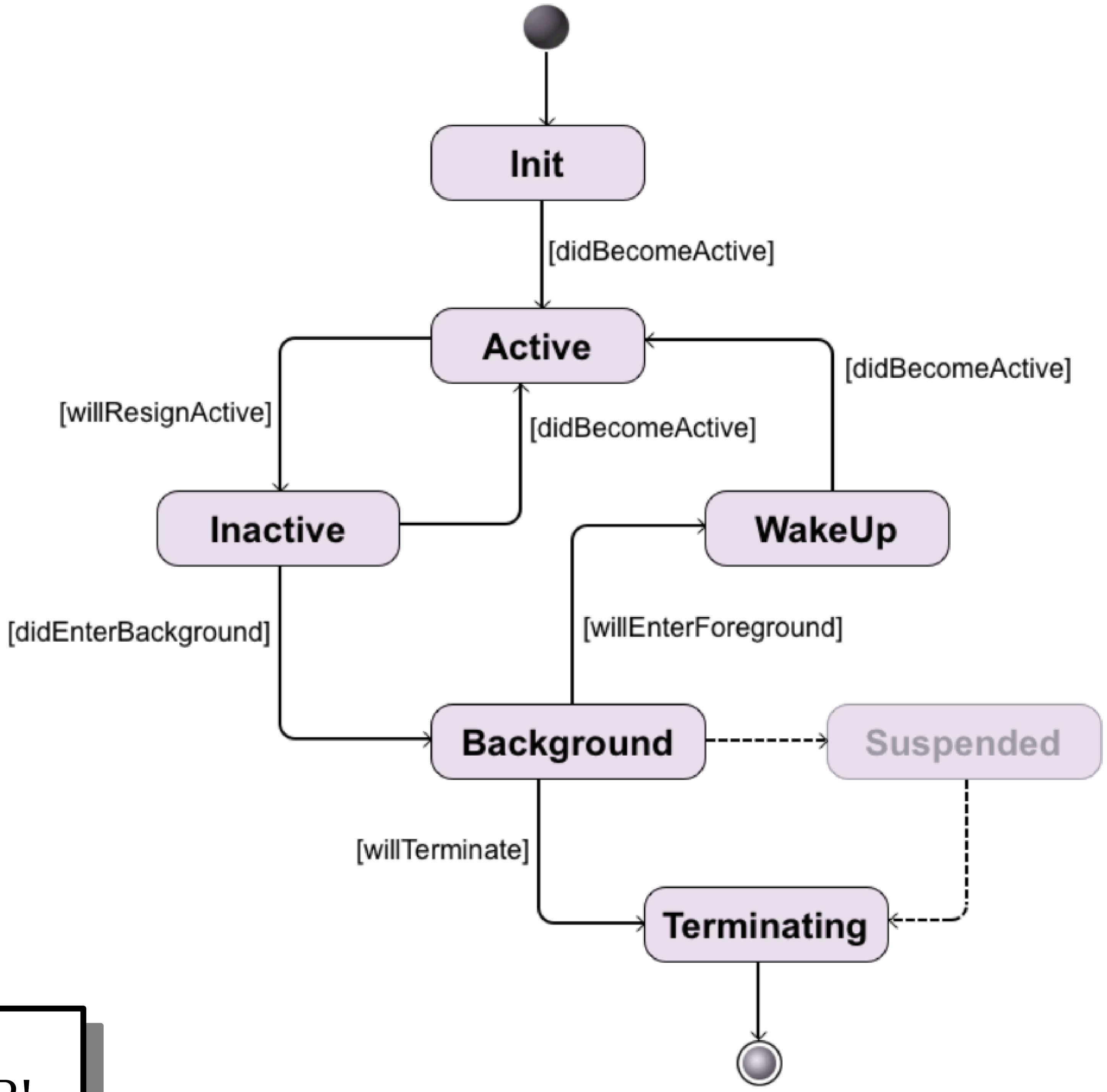
http://cs103.stanford.edu/tools/button-fsm/

Take CS148!



Take CS144!

http://www.tti.unipa.it/~gneglia/ip_networks06/slides/TCPIP_State_Transition_Diagram.pdf



Take
CS193P!

https://blog.codecentric.de/en/2016/07/handling-ios-app-states-state-machine/

Computers as Finite Automata

- My computer has 12GB of RAM and about 150GB of hard disk space.
- That's a total of 162GB of memory, which is 1,391,569,403,904 bits.
- There are “only” $2^{1,391,569,403,904}$ possible configurations of the memory in my computer.
- You could in principle build a DFA representing my computer, where there's one symbol per type of input the computer can receive.

A Powerful Intuition

- ***Regular languages correspond to problems that can be solved with finite memory.***
  - At each point in time, we only need to store one of finitely many pieces of information.
- Nonregular languages, in a sense, correspond to problems that cannot be solved with finite memory.
- Since every computer ever built has finite memory, in a sense, nonregular languages correspond to problems that cannot be solved by physical computers!

Finding Nonregular Languages

Finding Nonregular Languages

- To prove that a language **is regular**, we can just find a DFA, NFA, or regex for it.
- To prove that a language **is not regular**, we need to prove that there is **no possible** DFA for it.
  - *(or no possible NFA, or regex---but since these are equivalent we only need to show one is impossible)*
- ***This sort of argument will be challenging!***

Finding Nonregular Languages

- What kind of characteristics make a language too hard for any of these to handle?
  - Deterministic Finite Automata (DFA)
  - Nondeterministic Finite Automata (NFA)
  - Regular Expression

PollEv.com/cs103spr25:
T/F: You can't make a DFA for a language (set of strings) with infinite cardinality, so a language with infinite cardinality cannot be a regular langauge.

A Simple Language

- Let $\Sigma = \{$**a**, **b**$\}$ and consider the following language:

$$E = \{ \mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N} \}$$

- E is the language of all strings of n **a**'s followed by n **b**'s:

{ ε, **ab**, **aabb**, **aaabbb**, **aaaabbbb**, … }

A Simple Language

$$E = \{ \mathbf{a}^n \mathbf{b}^n \mid n \in \mathbb{N} \}$$

PollEv.com/cs103spr25: Which of the following are correct regular expressions for the language E defined above?

a*b*

(ab)*

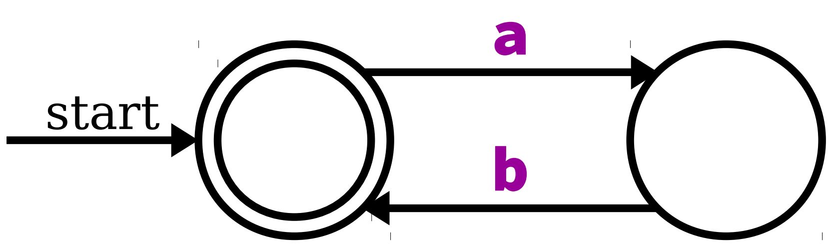
$\boldsymbol{\varepsilon} \cup \mathbf{ab} \cup \mathbf{a^2b^2} \cup \mathbf{a^3b^3}$

Another Attempt

- Let's try to design an NFA for

$$E = \{ \mathbf{a}^n \mathbf{b}^n \mid n \in \mathbb{N} \}.$$

- Does this machine work?



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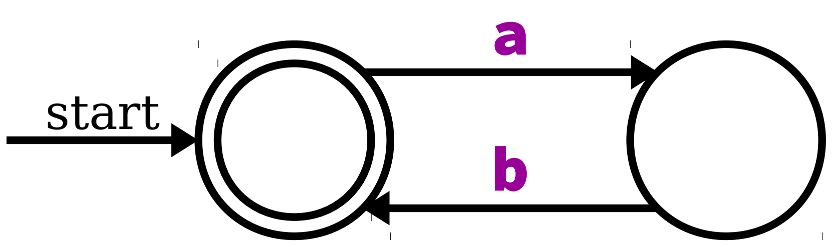
What is a regex that describes the language of this NFA?

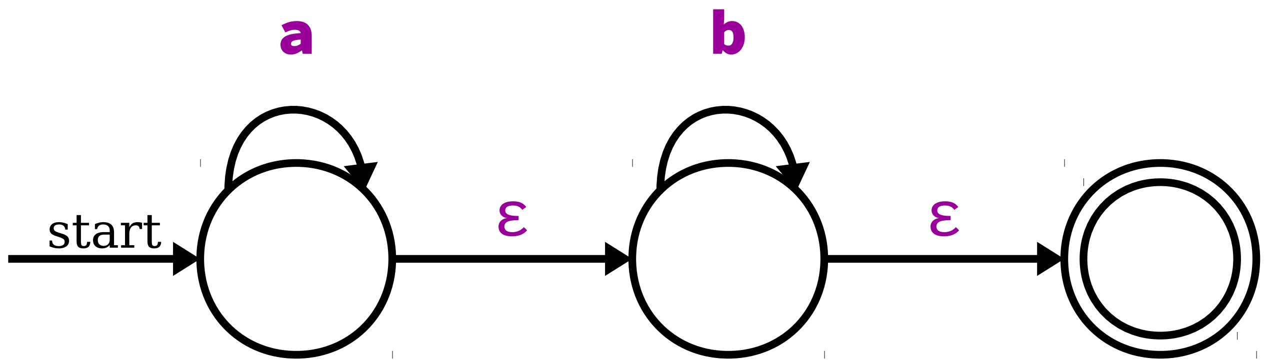


Another Attempt

- Let’s try to design an NFA for

$$E = \{ \mathbf{a}^n \mathbf{b}^n \mid n \in \mathbb{N} \}.$$

- How about this one?



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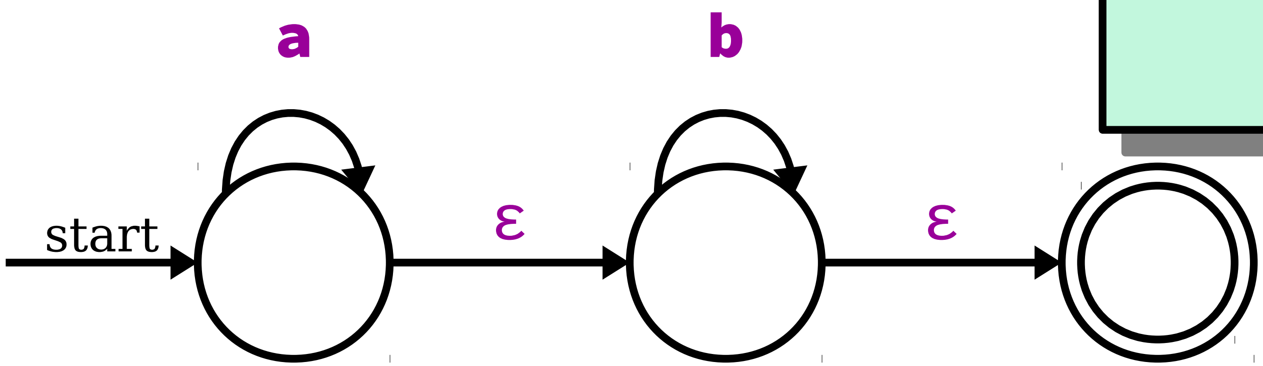

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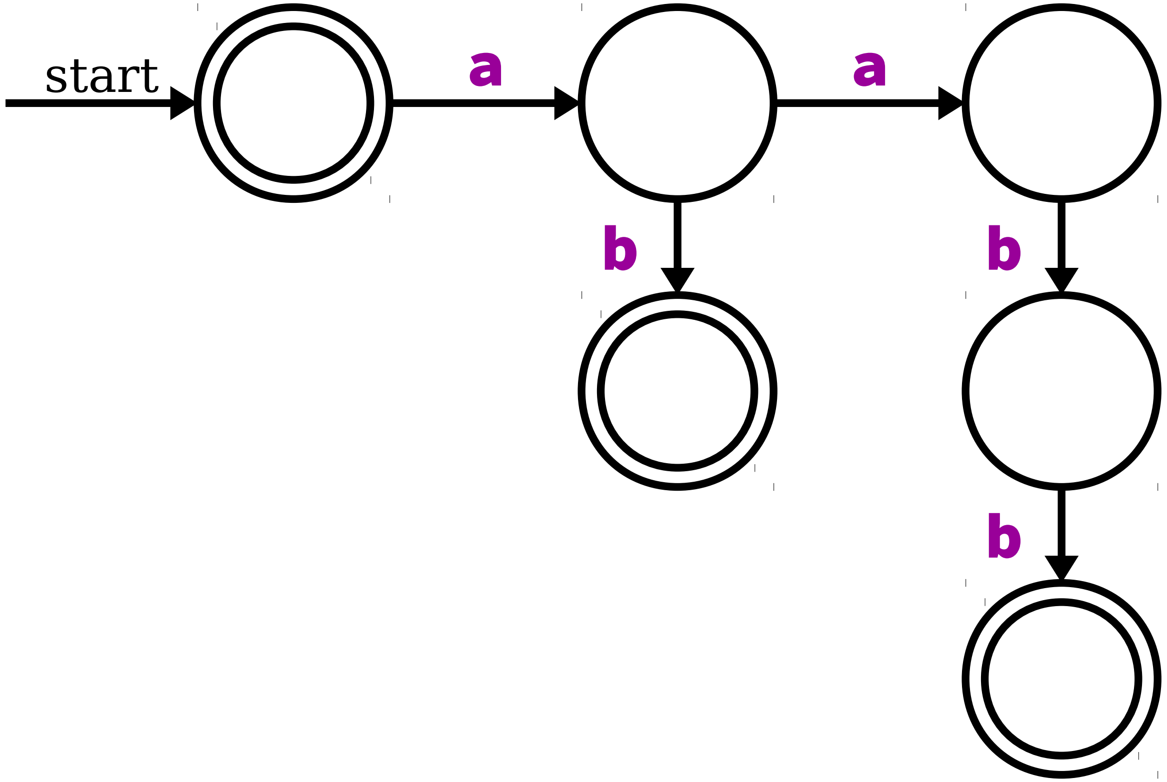

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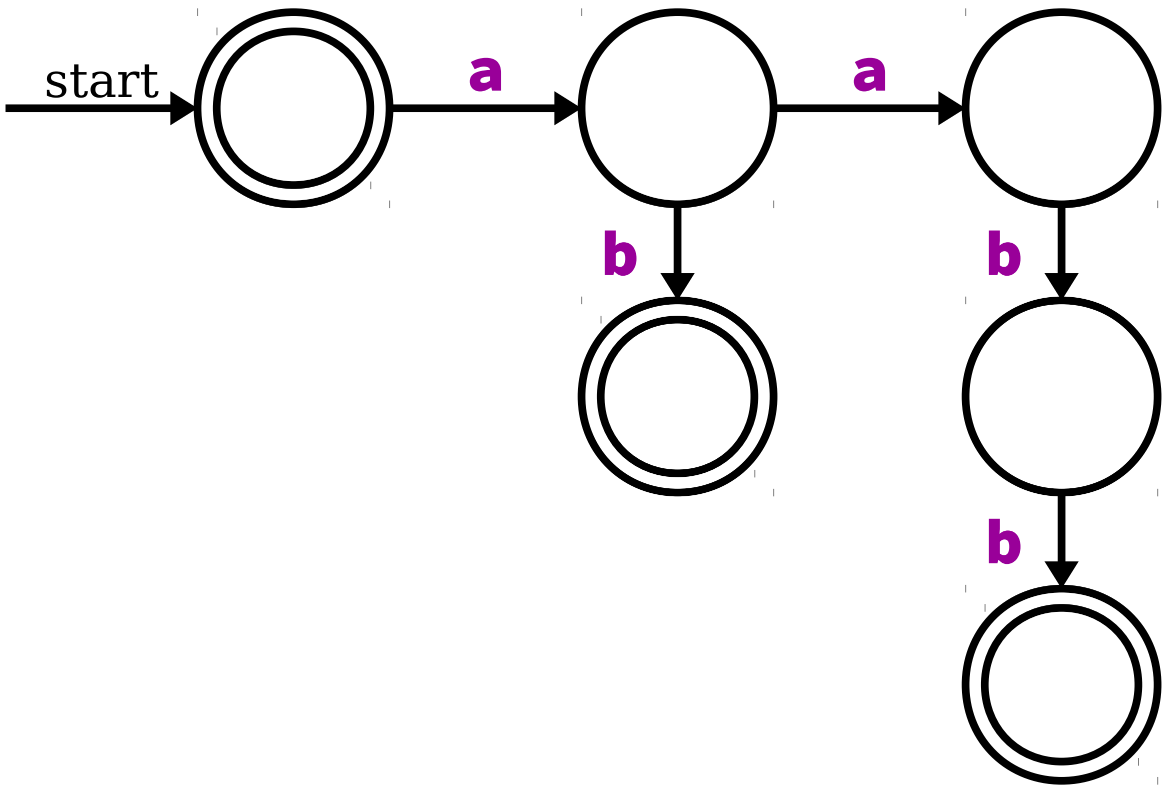
- What about this?

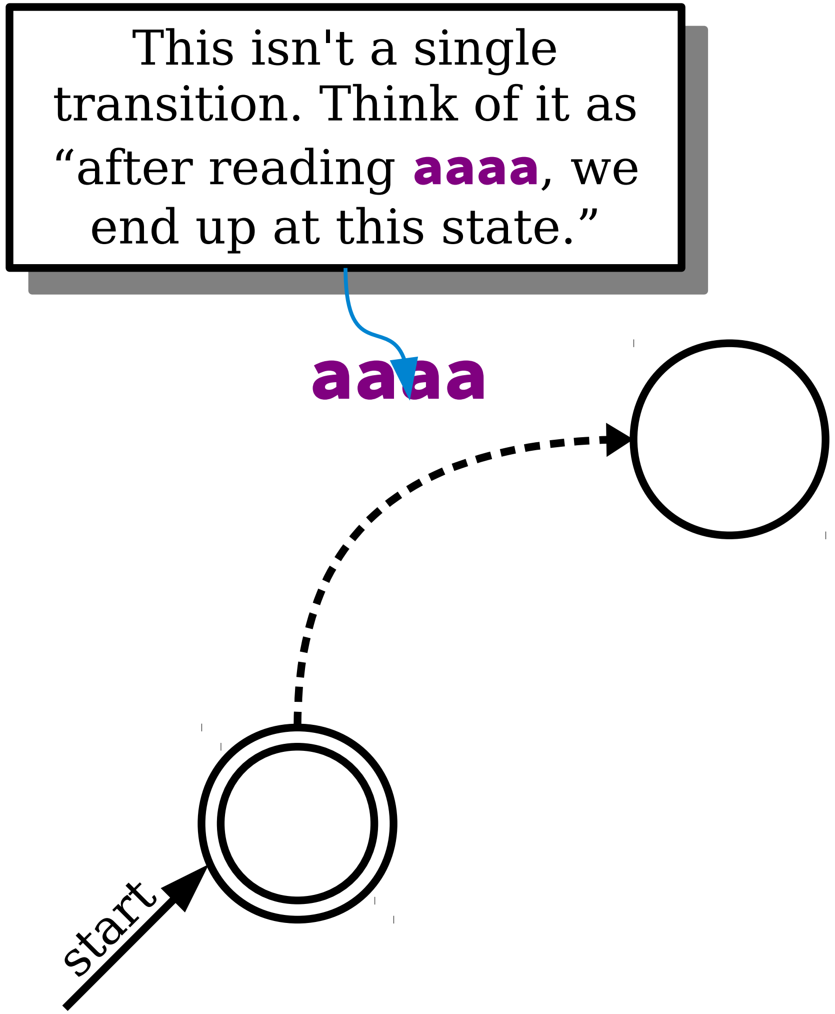


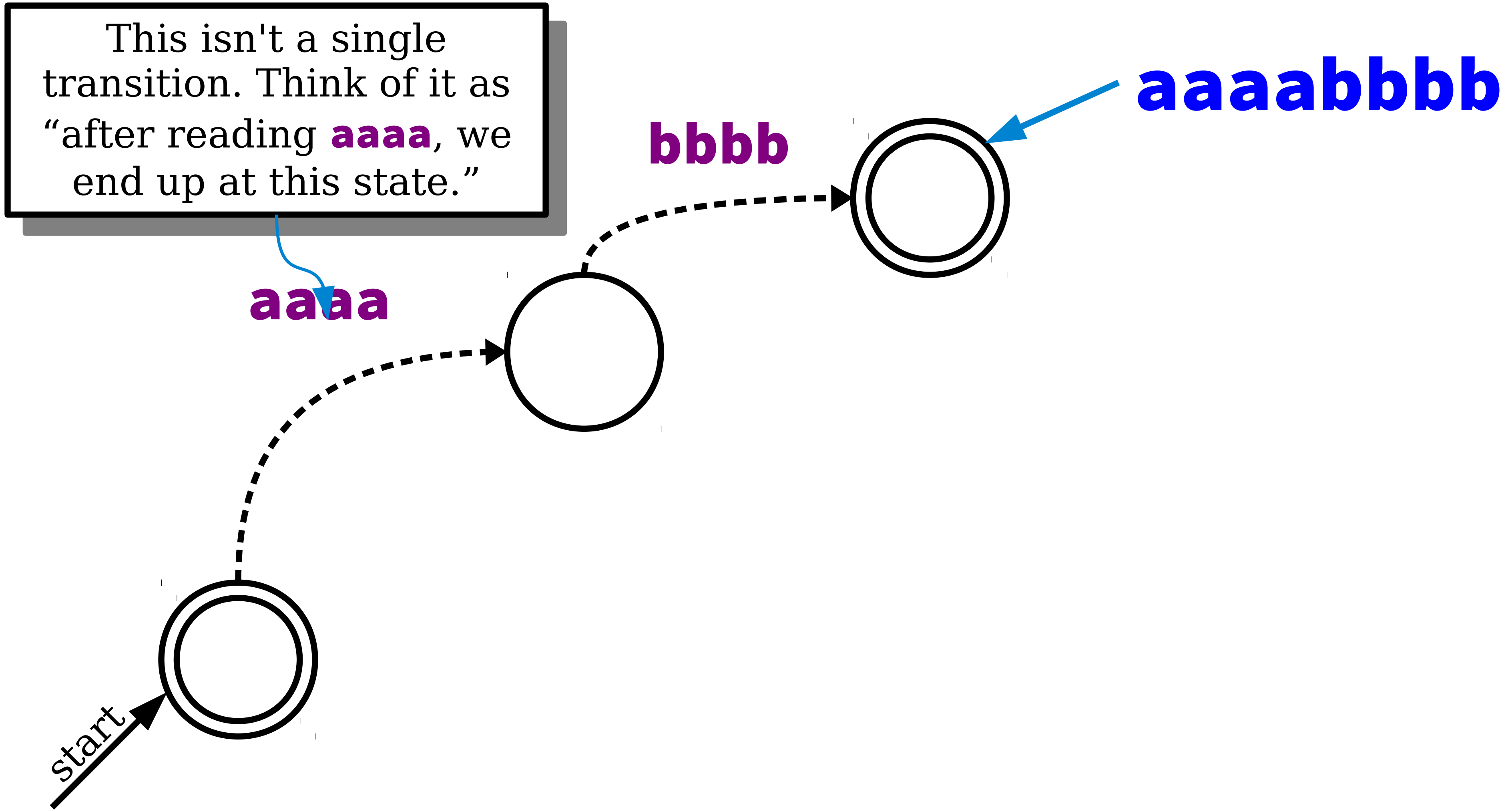
PollEv.com/cs103spr25:
What is a regex that describes the language of this NFA?

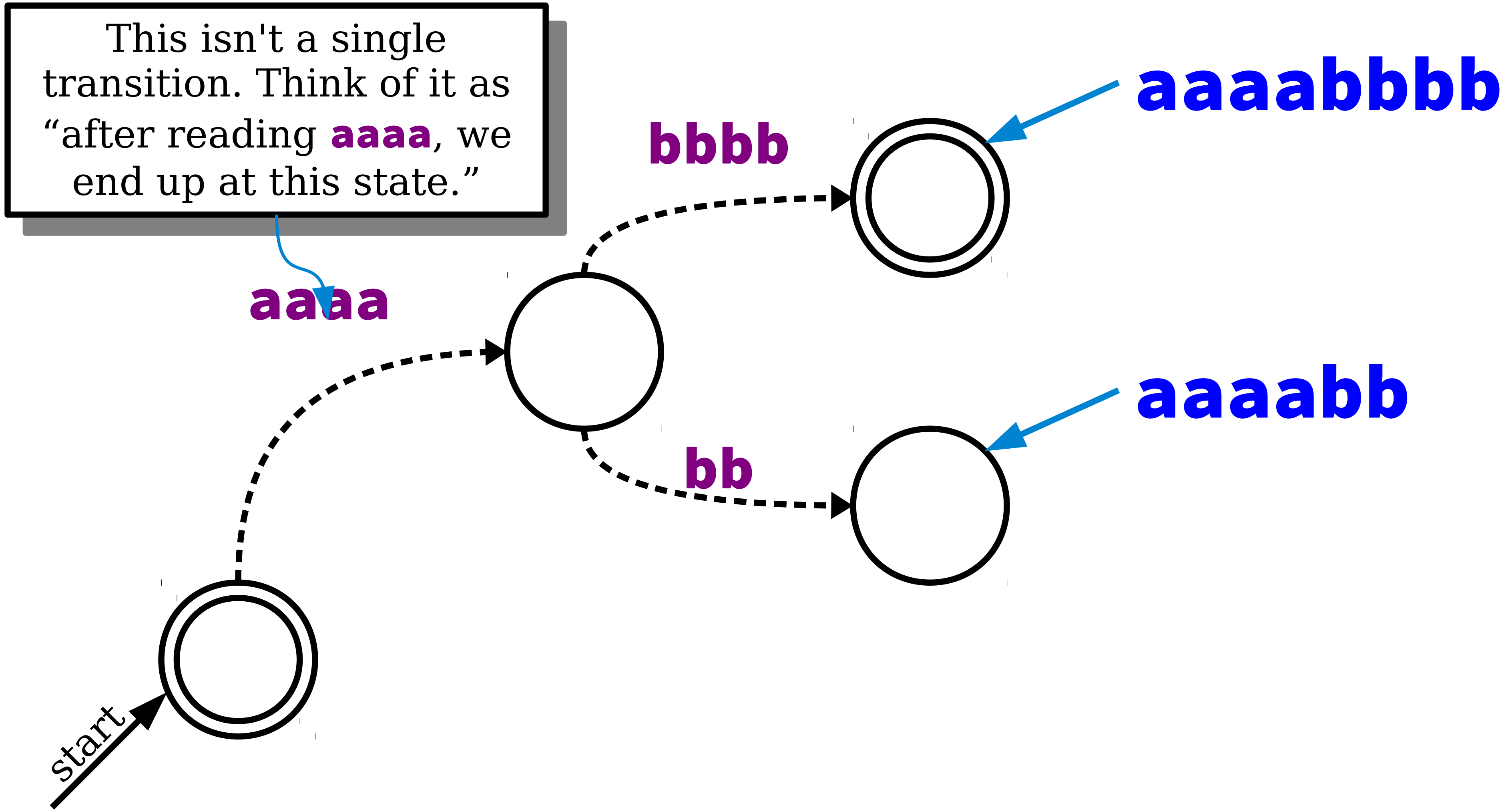
We seem to be running into some trouble.
Why is that?

Let's imagine what a DFA for the language $\{ \mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N}\}$ would have to look like.

Can we say anything about it?


This isn't a single transition. Think of it as "after reading **aaaa**, we end up at this state."
aaaa
start


This isn't a single transition. Think of it as "after reading aaaa, we end up at this state."
aaaabbbb
bbbb
aaaa
start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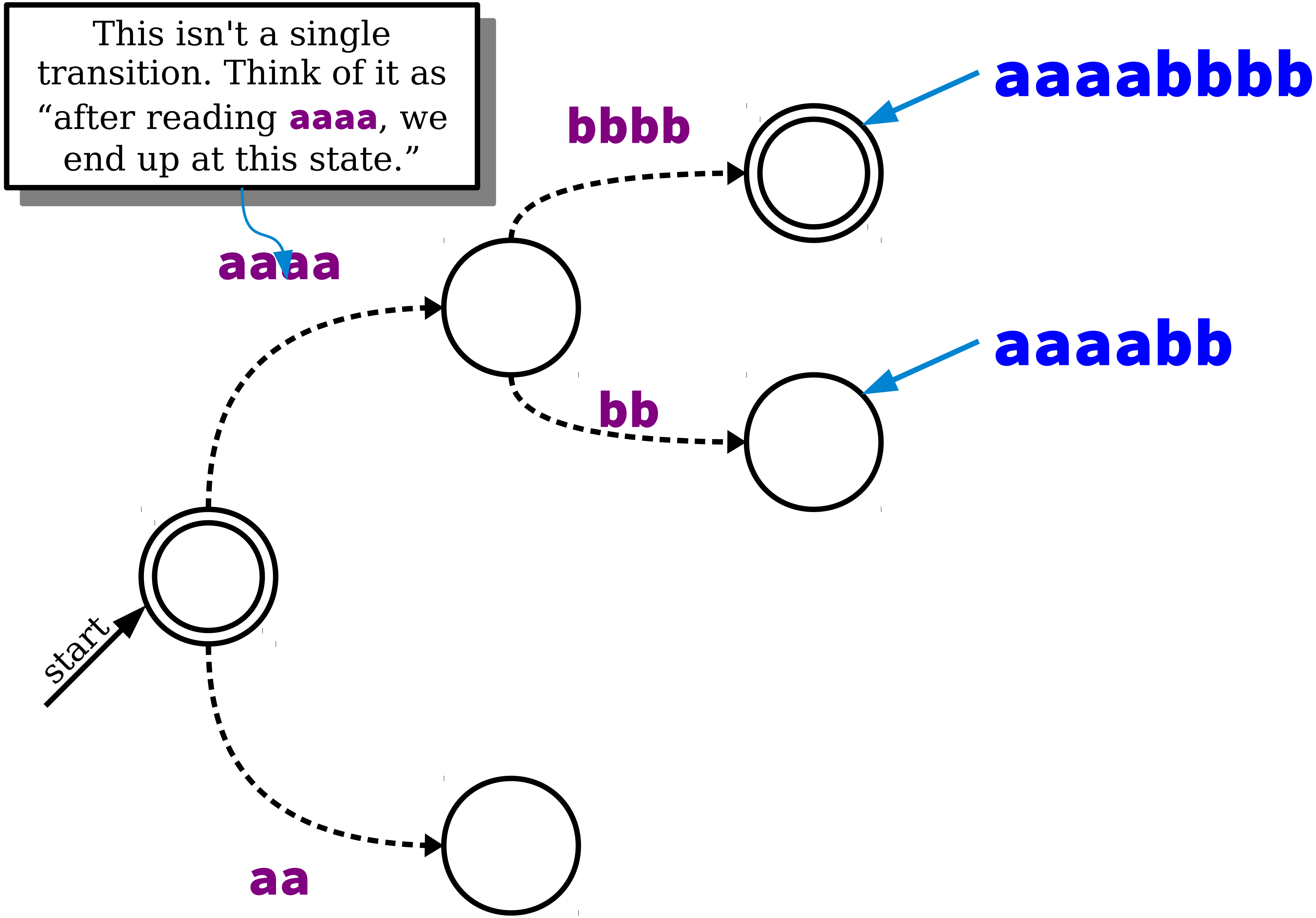
aaaa
bbbb
bb
aaaabbbb
aaaabb
start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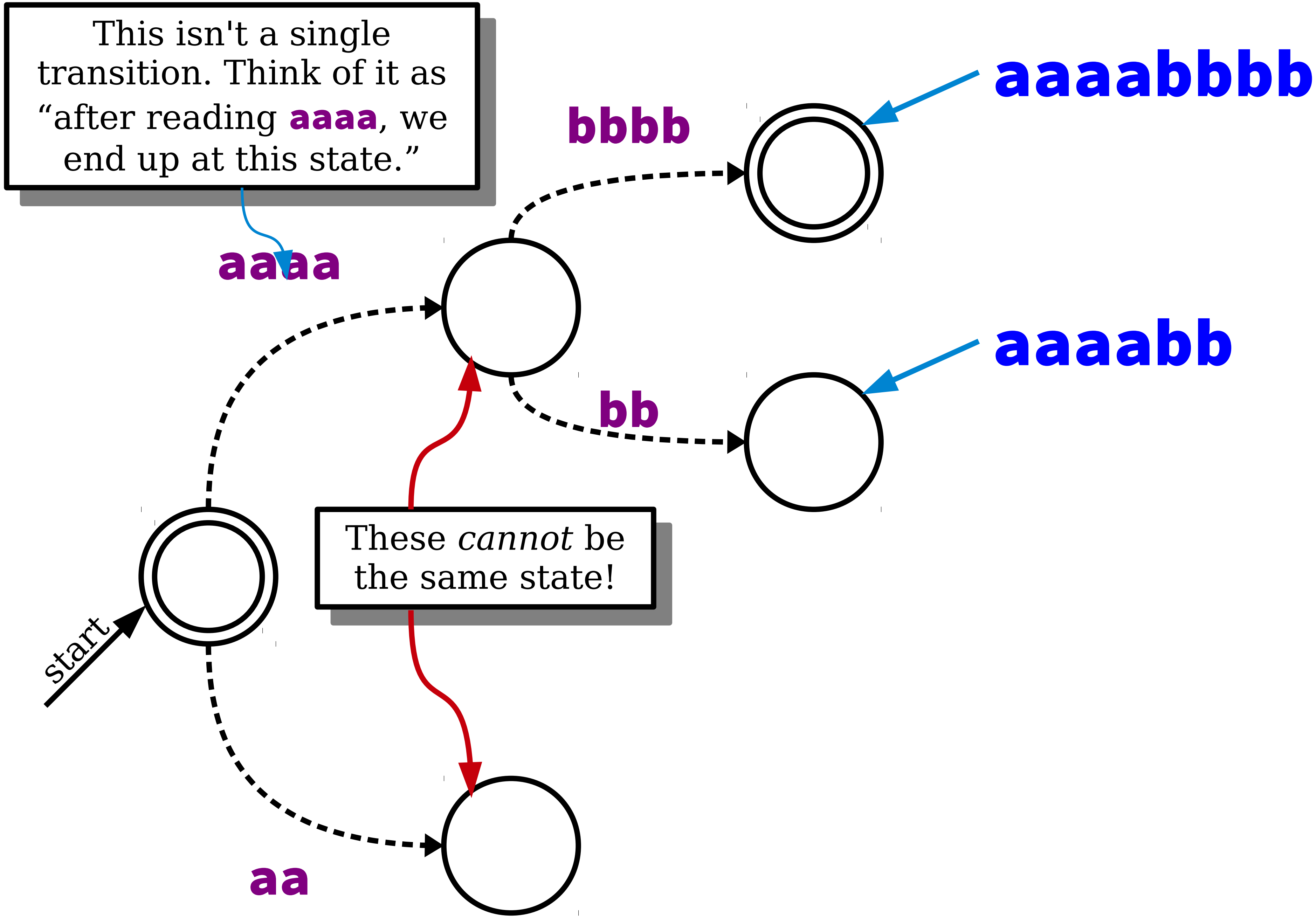
aaaabbbb
bbbb
aaaa
aaaabb
bb
start
aa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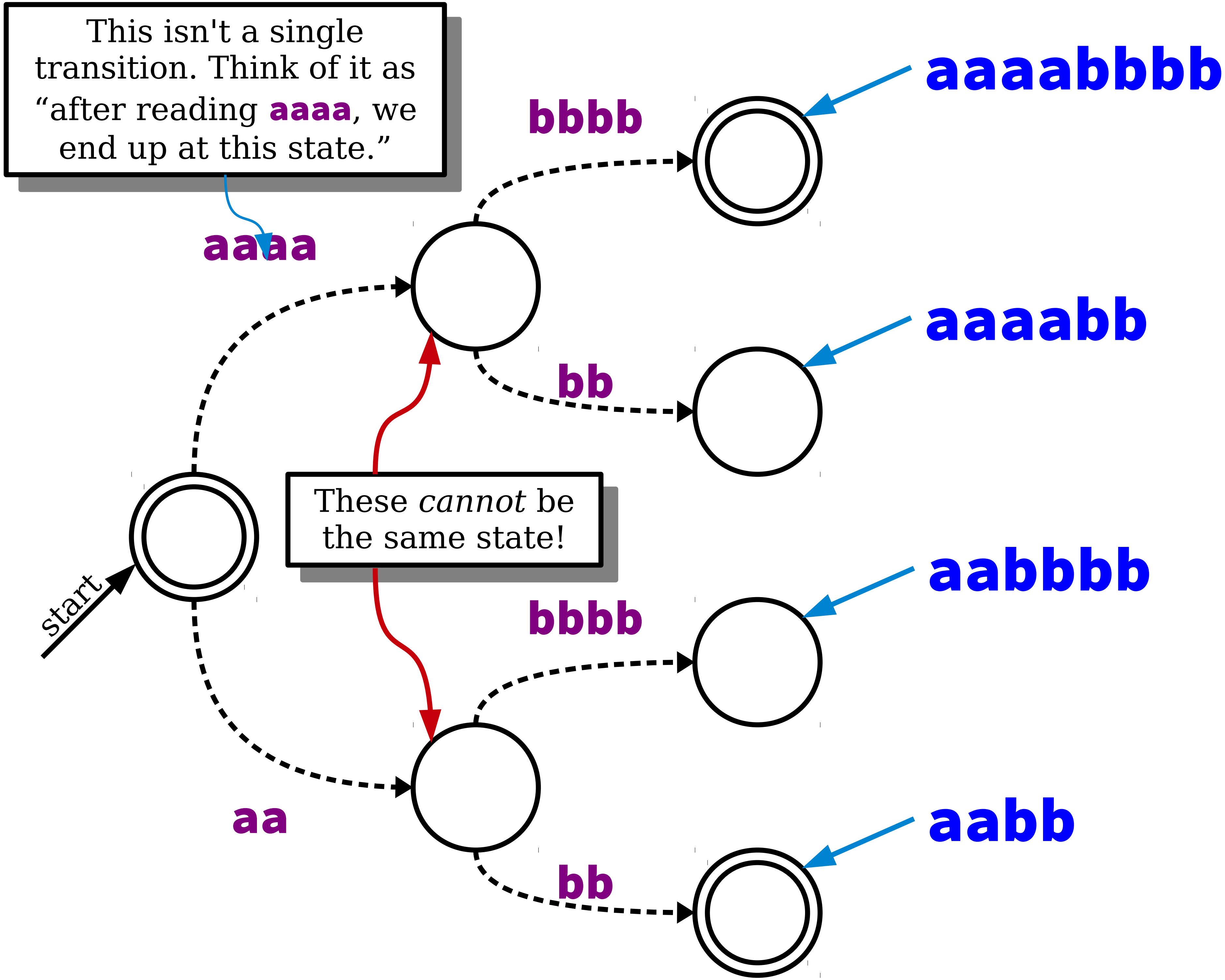
aaaa
bbbb
aaaabbbb
aaaabb
bb
start
These cannot be the same state!
aa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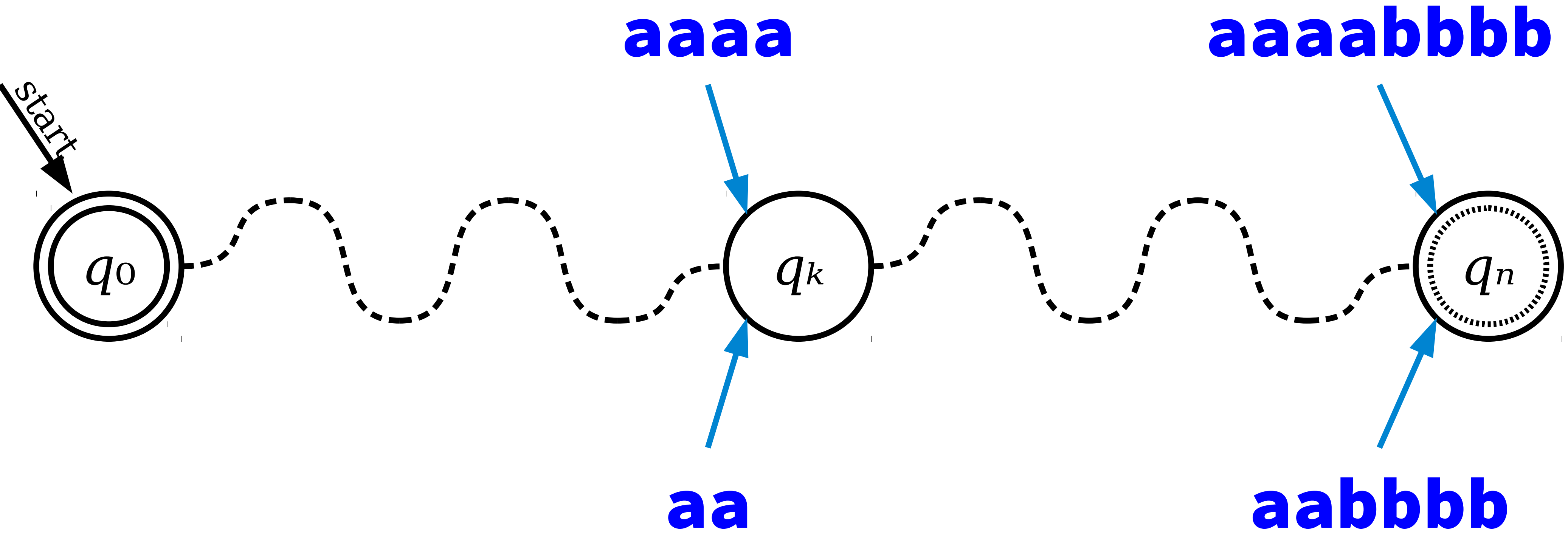
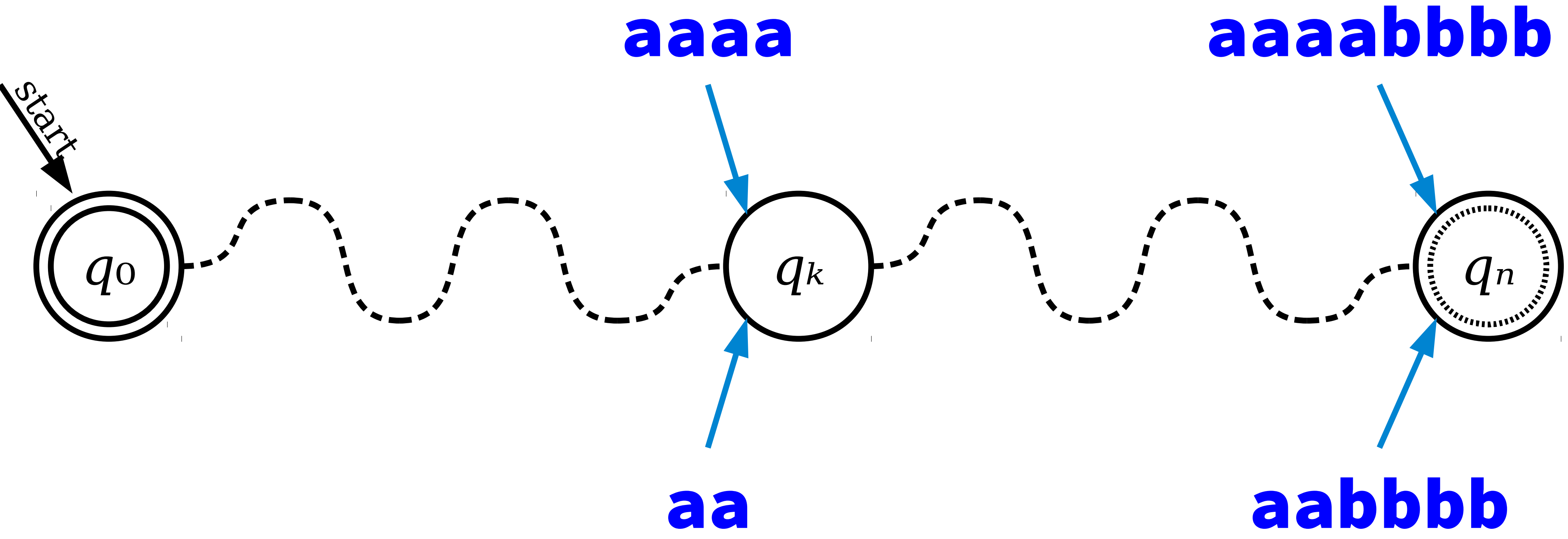
aaaa
bbbb
bb
aaaabbbb
aaaabb
These cannot be the same state!
start
aa
bbbb
bb
aabbbb
aabb

A Different Perspective



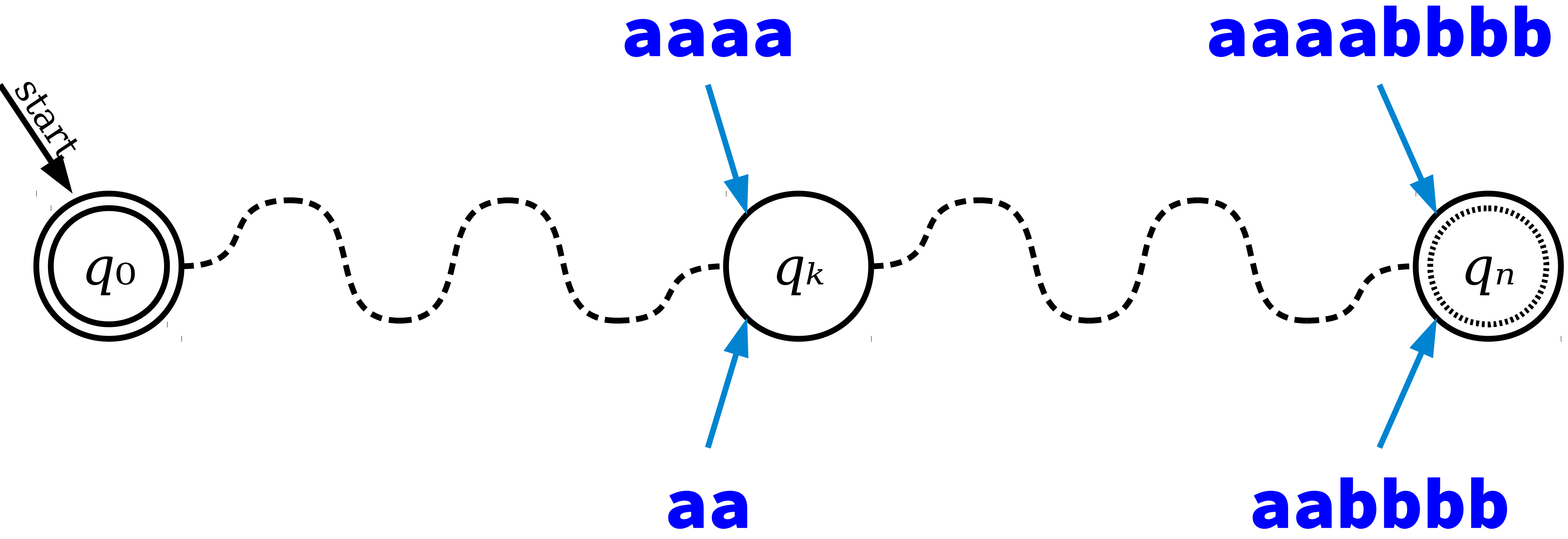
A Different Perspective



What happens if q_n is…

…an accepting state?
…a rejecting state?

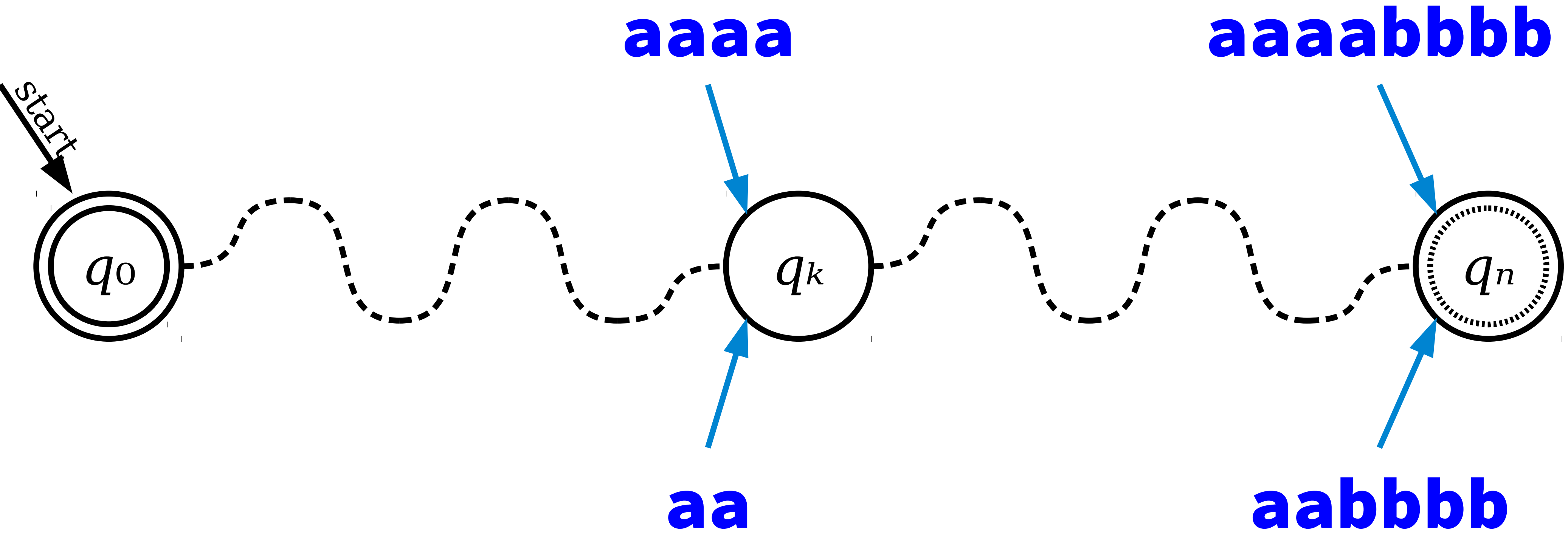
A Different Perspective



What happens if q_n is…

…an accepting state? We accept **aabbbb** $\notin E$!
…a rejecting state?

A Different Perspective



What happens if q_n is…

…an accepting state? We accept **aabbbb** $\notin E$!
…a rejecting state? We reject **aaaabbbb** $\in E$!

What’s Going On?

- As you just saw, the strings $\mathbf{a}^4$ and $\mathbf{a}^2$ can't end up in the same state in *any* DFA for $E = \{\mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N}\}$.
- Two proof routes:
  - *Direct:* The states you reach for $\mathbf{a}^4$ and $\mathbf{a}^2$ have to behave differently when reading $\mathbf{b}^4$ – in one case it should lead to an accept state, in the other it should lead to a reject state. Therefore, they must be different states.
  - *Contradiction:* Suppose you do end up in the same state. Then $\mathbf{a}^4\mathbf{b}^4$ and $\mathbf{a}^2\mathbf{b}^4$ end up in the same state, so we either reject $\mathbf{a}^4\mathbf{b}^4$ (oops) or accept $\mathbf{a}^2\mathbf{b}^4$ (oops).
- ***Powerful intuition:*** Any DFA for E must keep $\mathbf{a}^4$ and $\mathbf{a}^2$ separated. It needs to remember something fundamentally different after reading those strings.

This idea – that two strings shouldn't end up in the same DFA state – is fundamental to discovering nonregular languages.

Let's go formalize this!

Distinguishability

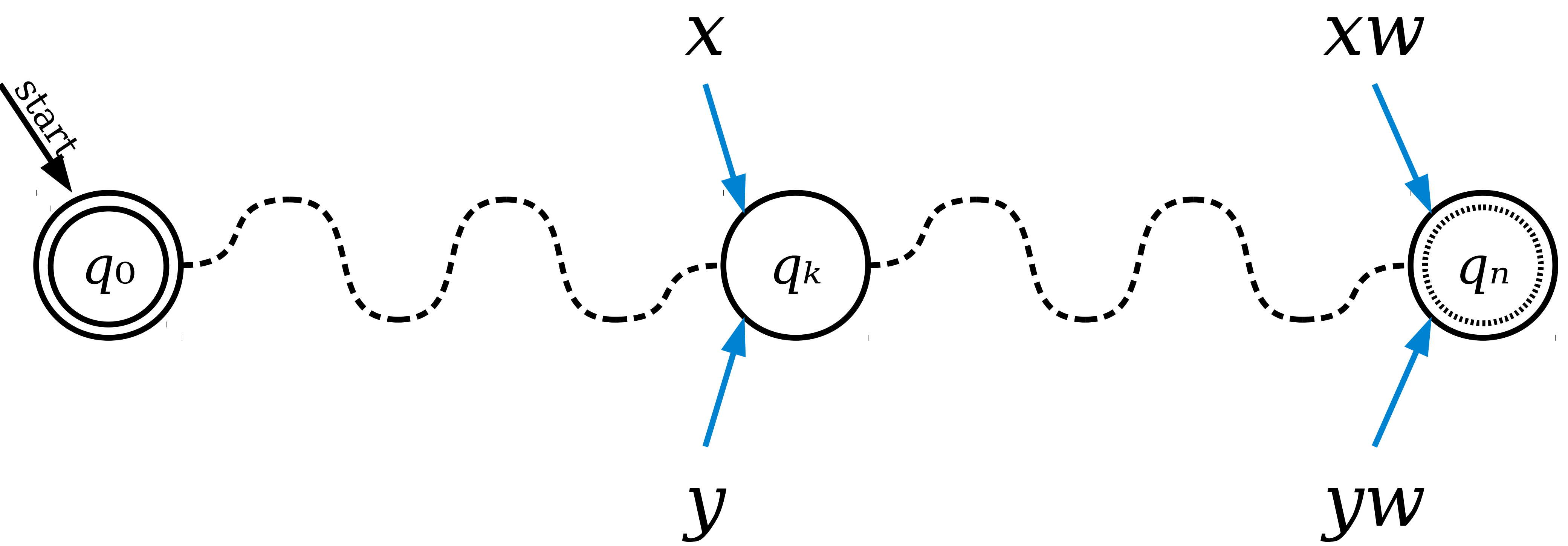
- Let L be an arbitrary language over Σ.
- Two strings $x \in \Sigma^*$ and $y \in \Sigma^*$ are called ***distinguishable relative to L*** if there is a string $w \in \Sigma^*$ such that exactly one of xw and yw is in L.
- We denote this by writing $x \not\equiv_L y$.
- In our previous example, we saw that $\mathbf{a}^2 \not\equiv_E \mathbf{a}^4$.
  - Try appending $\mathbf{b}^4$ to both of them.
- Formally, we say that $x \not\equiv_L y$ if the following is true:

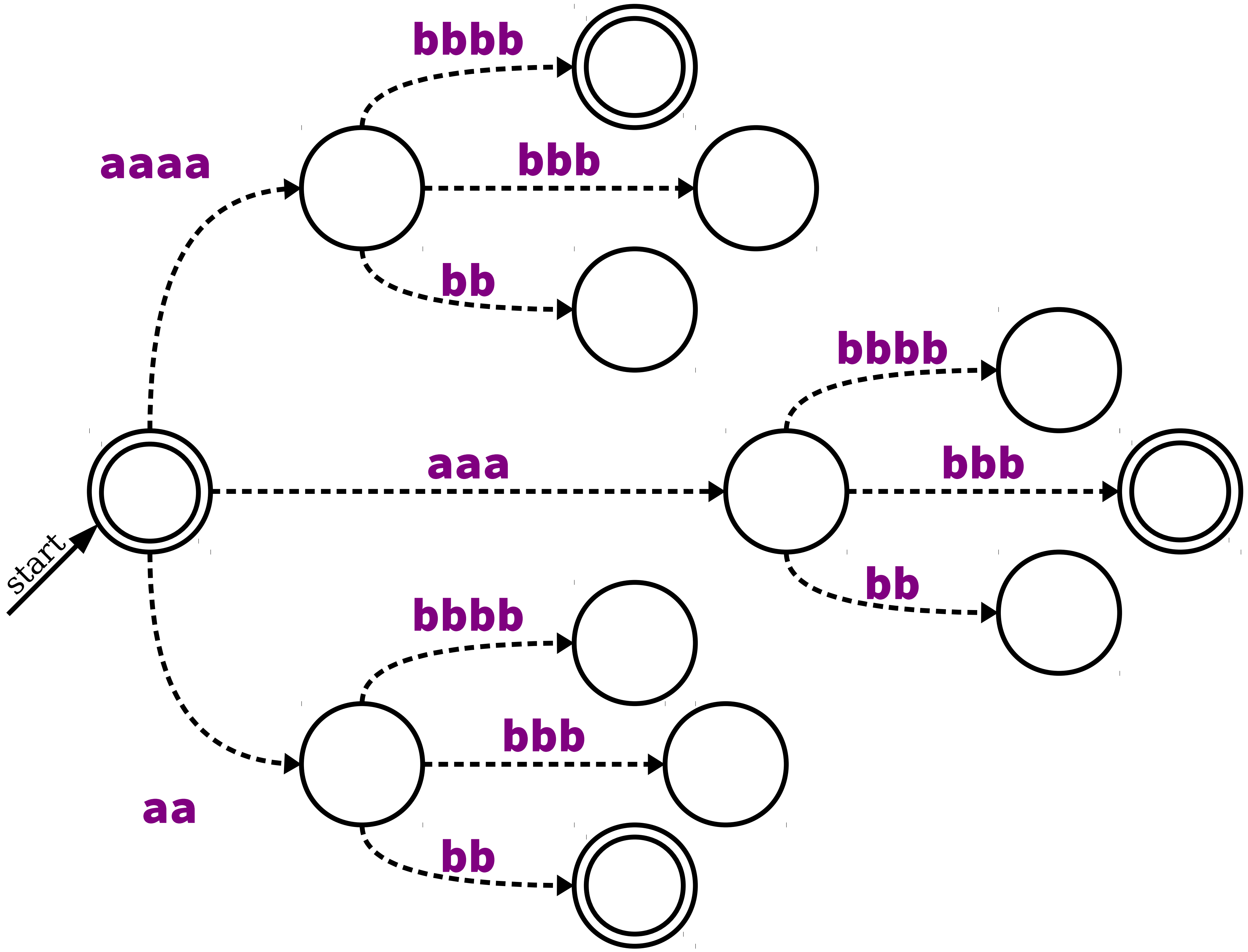
$$\exists w \in \Sigma^*. (xw \in L \leftrightarrow yw \notin L)$$

Write this down!

Distinguishability

- ***Theorem:*** Let L be an arbitrary language over Σ. Let $x \in \Sigma^*$ and $y \in \Sigma^*$ be strings where $x \not\equiv_L y$. Then if D is ***any*** DFA for L, then D must end in different states when run on inputs x and y.
- ***Proof sketch:***




bbbb
aaaa
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start
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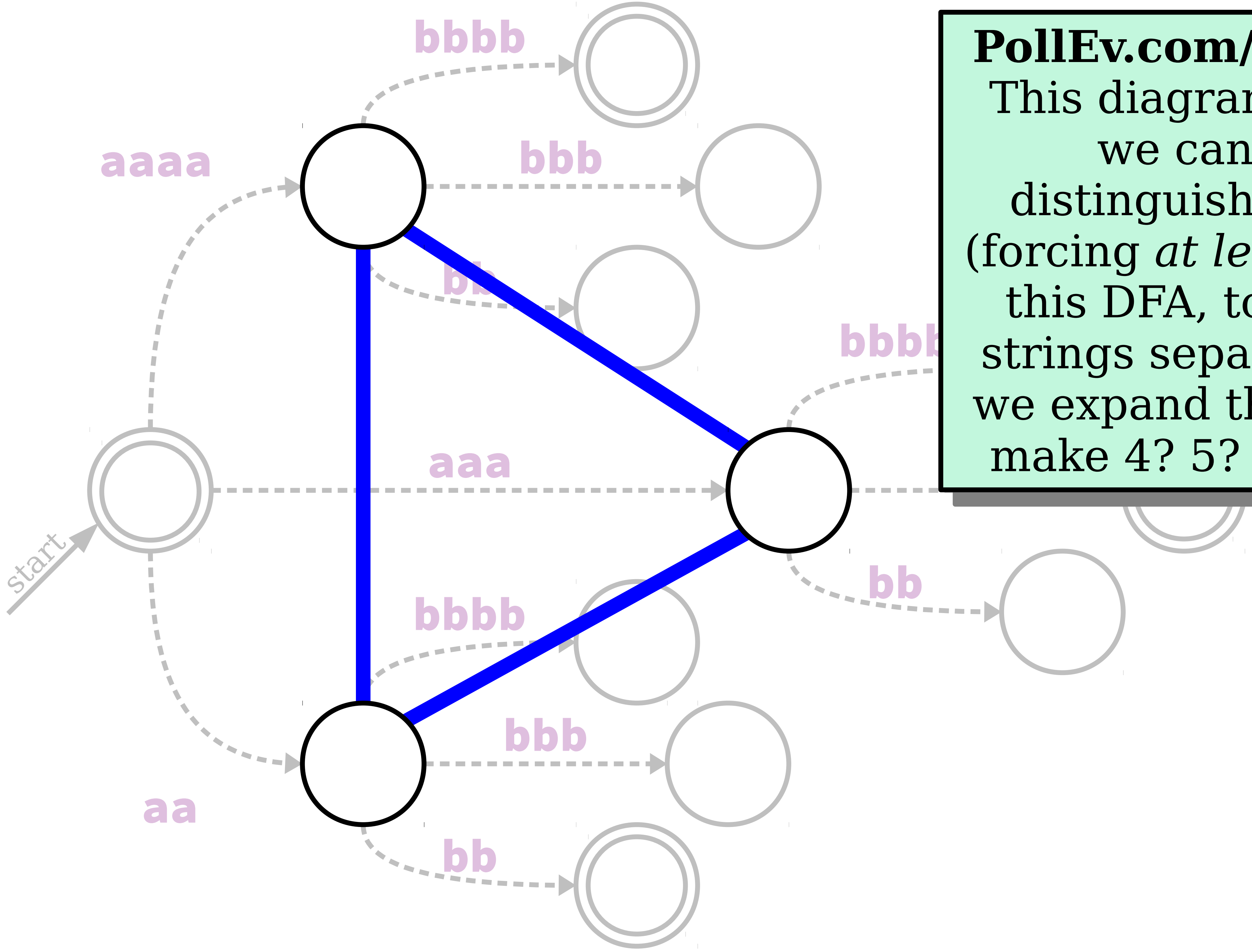
aa
bb



PollEv.com/cs103spr25: This diagram shows that we can make 3 distinguishable strings (forcing *at least* 3 states in this DFA, to keep these strings separated). Could we expand the diagram to make 4? 5? **How many?**

A Bad Combination

- Suppose there is a DFA D for the language $E = \{\mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N}\}$.
- We know the following:
  - Any two strings of the form $\mathbf{a}^m$ and $\mathbf{a}^n$, where $m \neq n$, cannot end in the same state when run through D.
  - There are **infinitely many** strings of the form $\mathbf{a}^m$.
  - However, there are only *finitely many* states they can end up in, since D is a deterministic ***finite*** automaton!
- What happens if we put these pieces together?

Distinguishing Sets

We say that ***x* $\not\equiv_L$ *y*** if the following is true:
$\exists w \in \Sigma^*. (xw \in L \leftrightarrow yw \notin L)$

- Let L be a language over Σ.
- A ***distinguishing set*** for L is a set $S \subseteq \Sigma^*$ where the following is true:

$$\forall x \in S.\ \forall y \in S.\ (x \neq y \rightarrow x \not\equiv_L y)$$

Distinguishing Sets

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$$\forall x \in S.\ \forall y \in S.\ (x \neq y \rightarrow x \not\equiv_L y)$$

If you pick any two distinct strings in *S*…

… then they're distinguishable relative to *L*.

Distinguishing Sets

> We say that $x \not\equiv_L y$ if the following is true:
> $\exists w \in \Sigma^*. (xw \in L \leftrightarrow yw \notin L)$

- Let L be a language over Σ.
- A ***distinguishing set*** for L is a set $S \subseteq \Sigma^*$ where the following is true:

$$\forall x \in S. \forall y \in S. (x \neq y \to x \not\equiv_L y)$$

- As an example, here's a distinguishing set for $E = \{ \mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N} \}$:

$$S = \{ \mathbf{a}^n \mid n \in \mathbb{N} \}$$

> **IMPORTANT:**
> A distinguishing set for the language E is **not a subset** of E. It is a set of **prefixes** (beginning part) of strings in E.

Theorem (Myhill-Nerode): If L is a language and S is a distinguishing set for L that contains infinitely many strings, then L is not regular.

Proof: Let L be an arbitrary language over Σ and let S be a distinguishing set for L that contains infinitely many strings. We will show that L is not regular.

Suppose for the sake of contradiction that L is regular. This means that there must be some DFA D for L. Let k be the number of states in D. Since there are infinitely many strings in S, we can choose $k+1$ distinct strings from S and consider what happens when we run D on all of those strings. Because there are only k states in D and we've chosen $k+1$ strings from S, by the pigeonhole principle we know that at least two strings from S must end in the same state in D. Choose any two such strings and call them x and y.

Because $x \neq y$ and S is a distinguishing set for L, we know that $x \not\equiv_L y$. Our earlier theorem therefore tells us that when we run D on inputs x and y, they must end up in different states. But this is impossible – we chose x and y precisely because they end in the same state when run through D.

We have reached a contradiction, so our assumption must have been wrong. Thus L is not a regular language. ■

Using the Myhill-Nerode Theorem for $E = \{ \mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N} \}$

Theorem: The language $E = \{\ \mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N}\ \}$ is not regular.

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To see that S is infinite, note that S contains one string for each natural number.

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Since S is infinite and is a distinguishing set for E, by the Myhill-Nerode theorem we see that E is not regular. ■

Myhill-Nerode Theorem: If L is a language and S is a distinguishing set for L that contains infinitely many strings, then L is not regular.

What Just Happened?

- ***We've just hit the limit of finite-memory computation.***
- To build a DFA for $E = \{ \mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N} \}$, we need to have different memory configurations (states) for all possible strings of the form $\mathbf{a}^n$.
- There's no way to do this with finitely many possible states!

More Nonregular Languages

Another Language

- Consider the following language EQ over the alphabet $\Sigma = \{\mathbf{a}, \mathbf{b}, \stackrel{?}{=}\}$:

$$EQ = \{\ w \stackrel{?}{=} w \mid w \in \{\mathbf{a}, \mathbf{b}\}^* \}$$

- EQ is the language all strings consisting of the same string of **a**'s and **b**'s twice, with a $\stackrel{?}{=}$ symbol in-between.
- Examples:

$\mathbf{ab} \stackrel{?}{=} \mathbf{ab} \in EQ$ $\mathbf{bbb} \stackrel{?}{=} \mathbf{bbb} \in EQ$ $\stackrel{?}{=} \in EQ$

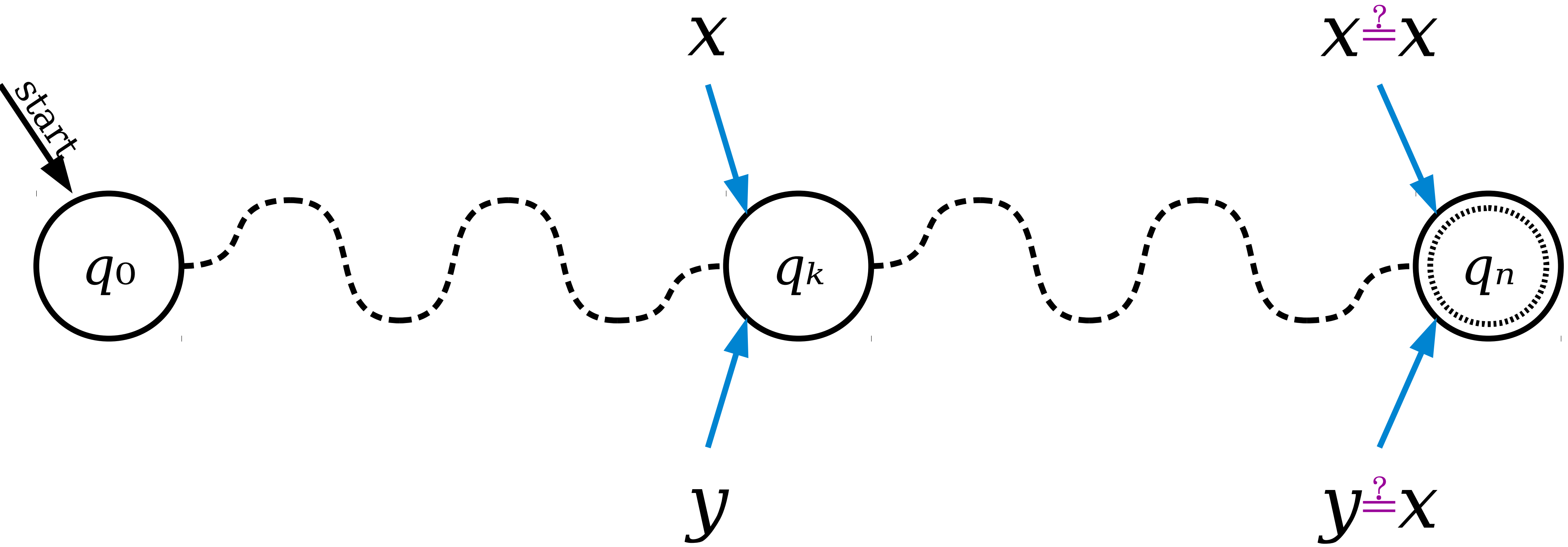
$\mathbf{ab} \stackrel{?}{=} \mathbf{ba} \notin EQ$ $\mathbf{bbb} \stackrel{?}{=} \mathbf{aaa} \notin EQ$ $\mathbf{b} \stackrel{?}{=} \notin EQ$

The Intuition

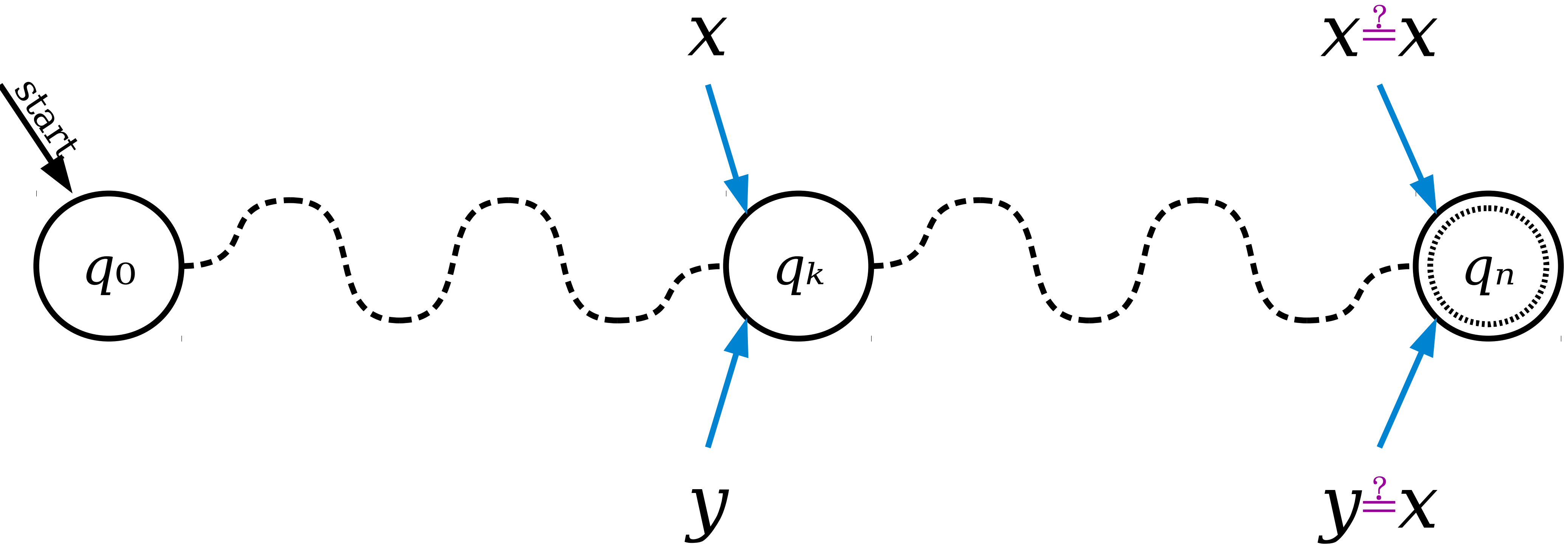
$$EQ = \{\ w \stackrel{?}{=} w \mid w \in \{\mathbf{a}, \mathbf{b}\}^* \}$$

- Intuitively, any machine for EQ has to be able to remember the contents of everything to the left of the $\stackrel{?}{=}$ so that it can match them against the contents of the string to the right of the $\stackrel{?}{=}$.
- There are infinitely many possible strings we can see, but we only have finite memory to store which string we saw.
- That's a problem... can we formalize this?

The Intuition



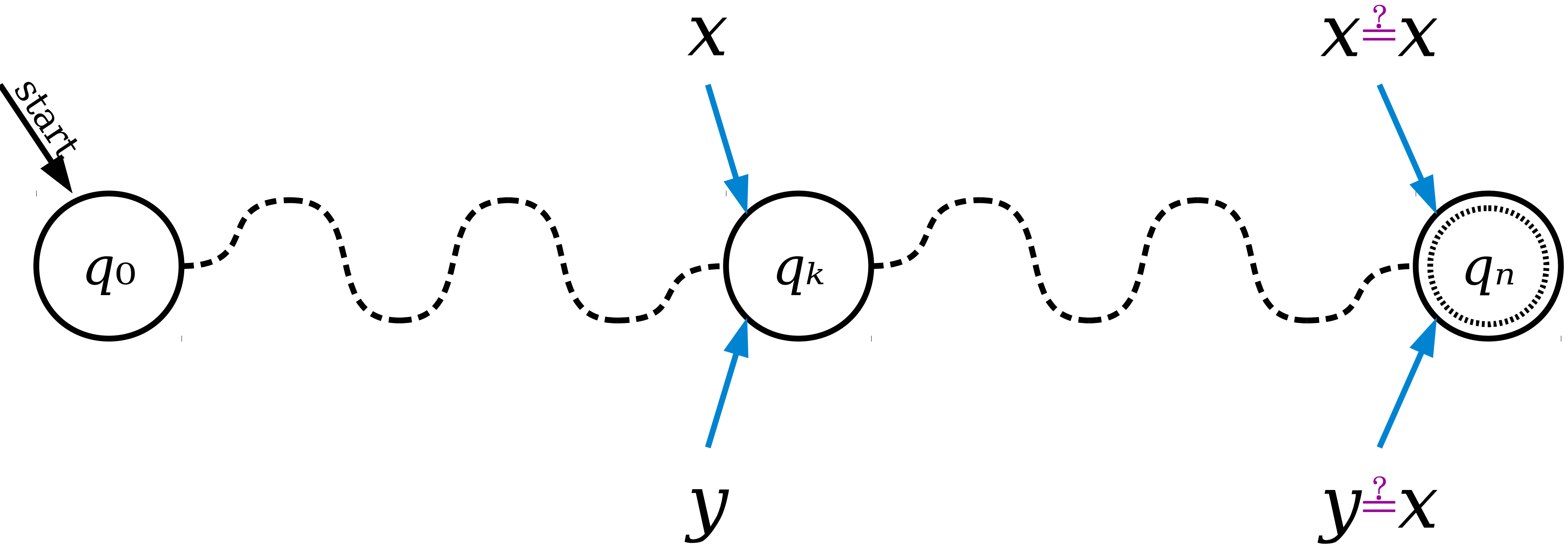
The Intuition



What happens if q_n is…

…an accepting state?
…a rejecting state?

The Intuition



What happens if q_n is…

…an accepting state? We accept $y \stackrel{?}{=} x \notin EQ$!

…a rejecting state?

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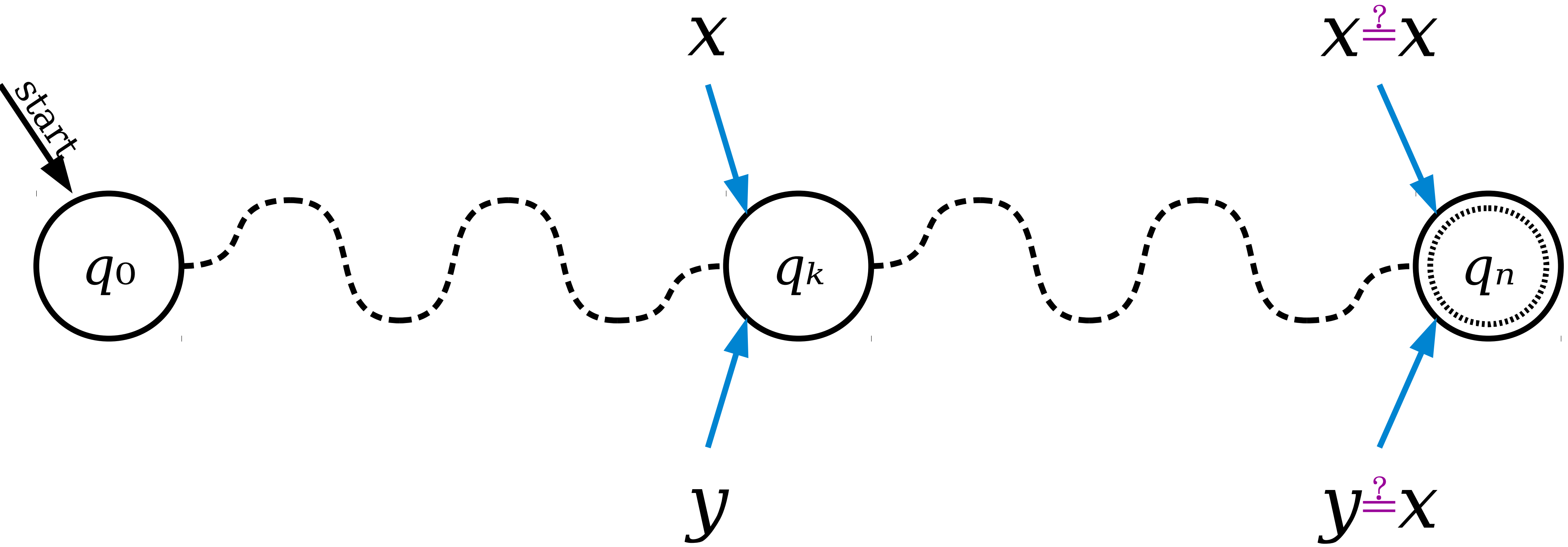

What happens if q_n is…

…an accepting state? We accept $y \stackrel{?}{=} x \notin EQ$!

…a rejecting state? We reject $x \stackrel{?}{=} x \in EQ$!

Another Language

- Consider the following language E the alphabet $\Sigma = \{$**a**, **b**, $\stackrel{?}{=}\}$:

$$EQ = \{\ w\stackrel{?}{=}w \mid w \in \{\mathbf{a}, \mathbf{b}\}^*\}$$

- EQ is the language all strings cons the same string of **a**'s and **b**'s twice $\stackrel{?}{=}$ symbol in-between.
- Examples:

ab≟ab $\in EQ$ **bbb≟bbb** $\in EQ$ ≟ $\in EQ$

ab≟ba $\notin EQ$ **bbb≟aaa** $\notin EQ$ **b≟** $\notin$

PollEv.com/cs103spr25:
Which of these are good infinite distinguishing sets for EQ that we could use in a Myhill-Nerode proof?

1. $\{\ \mathbf{a}^n \mid n \in \mathbb{N}\ \}$
2. $\{\ \mathbf{b}^n \mid n \in \mathbb{N}\ \}$
3. $\{\mathbf{a}, \mathbf{b}\}^*$
4. $\{\ \mathbf{a}^n\stackrel{?}{=}\mathbf{a}^n \mid n \in \mathbb{N}\ \}$
5. $\{\ \mathbf{a}^n\stackrel{?}{=} \mid n \in \mathbb{N}\ \}$

We say that $x \not\equiv_L y$ if the following is true:

$\exists w \in \Sigma^*. (xw \in L \leftrightarrow yw \notin L)$

Distinguishing Sets

- We've identified a distinguishing set for $EQ = \{ w \stackrel{?}{=} w \mid w \in \{\mathtt{a}, \mathtt{b}\}^* \}$:

$$S = \{\mathtt{a}, \mathtt{b}\}^*$$

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IMPORTANT:
A distinguishing set for the language *EQ* is **not a subset** of *EQ*. It is a set of **prefixes** (beginning part) of strings in *EQ*.

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Theorem: The language $EQ = \{ w \stackrel{?}{=} w \mid w \in \{\mathbf{a}, \mathbf{b}\}^* \}$ is not regular.

Myhill-Nerode Theorem: If L is a language and S is a distinguishing set for L that contains infinitely many strings, then L is not regular.

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To see that S is infinite, note that, for any $n \in \mathbb{N}$, we have $\texttt{a}^n \in S$.

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Since S is infinite and a distinguishing set for EQ, by the Myhill-Nerode theorem we see that EQ is not regular, as required.

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Approaching Myhill-Nerode

- The challenge in using the Myhill-Nerode theorem is finding the right set of strings.
- ***General intuition:***
  - Start by thinking about what information a computer "must" remember in order to answer correctly.
  - Choose a group of strings that all require different information.
  - Prove that you have infinitely many strings an that the group of strings is a distinguishing set.

Another Language

- Consider the following language P over the alphabet $\Sigma = \{$**a**, **b**$\}$:

$$P = \{\ w \mid w \text{ is a palindrome}\}$$

- P is the language all strings where the second half is a "mirror" (reverse order) copy of the first half.
- Examples:

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2. $\{$ **b**$^n \mid n \in \mathbb{N} \}$
3. $\{$**a**, **b**$\}$*
4. $\{$ **a**^{n}**b** $\mid n \in \mathbb{N} \}$
5. $\{$ **a**^{n}**b**$^n \mid n \in \mathbb{N} \}$

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Tying Everything Together

- One of the intuitions we hope you develop for DFAs is to have each state in a DFA represent some key piece of information the automaton has to remember.
- If you only need to remember one of finitely many pieces of information, that gives you a DFA.
  - This can be made rigorous! Take CS154 for details.
- If you need to remember one of infinitely many pieces of information, you can use the Myhill-Nerode theorem to prove that the language has no DFA.

Where We Stand

Where We Stand

- We've ended up where we are now by trying to answer the question “what problems can you solve with a computer?”
- We defined a computer to be DFA, which means that the problems we can solve are precisely the regular languages.
- We've discovered several equivalent ways to think about regular languages (DFAs, NFAs, and regular expressions) and used that to reason about the regular languages.
- We now have a powerful intuition for where we ended up: DFAs are finite-memory computers, and regular languages correspond to problems solvable with finite memory.
- Putting all of this together, we have a much deeper sense for what finite memory computation looks like – *and what it doesn't look like!*

Where We're Going

- What does computation look like with unbounded memory?
- What problems can you solve with unbounded-memory computers?
- What does it even mean to “solve” such a problem?
- And how do we know the answers to any of these questions?

Next Time

- ***Context-Free Languages***
  - Context-Free Grammars
  - Generating Languages from Scratch